

## Appendix A: Derivation of Basic MM5 Equations

### A.1 Derivation of Thermodynamic Equation

First Law of Thermodynamics

$$dQ = c_v dT + p d\alpha = c_p dT - \alpha dp \quad (\text{A.1})$$

since from the gas law  $RdT = p d\alpha + \alpha dp$  and  $c_p - c_v = R$ . Temperature tendency therefore is given by

$$c_p \frac{DT}{Dt} = \frac{1}{\rho} \frac{Dp}{Dt} + \dot{Q} \quad (\text{A.2})$$

### A.2 Derivation of Pressure Tendency Equation

From Gas Law

$$\frac{1}{p} \frac{Dp}{Dt} = \frac{1}{\rho} \frac{D\rho}{Dt} + \frac{1}{T} \frac{DT}{Dt} \quad (\text{A.3})$$

Continuity and Thermodynamics lead to

$$\frac{1}{p} \frac{Dp}{Dt} = -\nabla \cdot \mathbf{v} + \frac{\dot{Q}}{c_p T} + \frac{1}{c_p \rho T} \frac{Dp}{Dt} \quad (\text{A.4})$$

However,  $c_p \rho T = \left(\frac{c_p}{R}\right)p$ , so

$$\frac{1}{p} \frac{Dp}{Dt} \left(1 - \frac{R}{c_p}\right) = -\nabla \cdot \mathbf{v} + \frac{\dot{Q}}{c_p T} \quad (\text{A.5})$$

But  $1 - \frac{R}{c_p} = \frac{c_v}{c_p} = \frac{1}{\gamma}$ , therefore

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + \frac{\gamma p \dot{Q}}{c_p T} \quad (\text{A.6})$$

### A.3 Forms of the Vertical Momentum Equation

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = D_w \quad (\text{A.7})$$

Defining  $\alpha = \frac{1}{\rho}$ ,

$$\frac{Dw}{Dt} + \alpha \frac{\partial p}{\partial z} + g = D_w \quad (\text{A.8})$$

Defining hydrostatic reference and perturbation,  $\alpha = \alpha_0 + \alpha'$ ,  $p = p_0 + p'$ ,

$$\frac{Dw}{Dt} + (\alpha_0 + \alpha') \left( \frac{\partial p_0}{\partial z} + \frac{\partial p'}{\partial z} \right) + g = D_w \quad (\text{A.9})$$

By definition,  $\alpha_0 \frac{\partial p_0}{\partial z} = -g$ , so

$$\frac{Dw}{Dt} + \alpha' \frac{\partial p'}{\partial z} + \alpha_0 \frac{\partial p'}{\partial z} + \alpha' \frac{\partial p_0}{\partial z} = D_w \quad (\text{A.10})$$

which can be written as

$$\frac{Dw}{Dt} + \alpha \frac{\partial p'}{\partial z} - g \frac{\alpha'}{\alpha_0} = D_w \quad (\text{A.11})$$

This can be expanded as

$$\frac{Dw}{Dt} + \alpha \frac{\partial p'}{\partial z} - g \frac{\alpha - \alpha_0}{\alpha_0} = D_w \quad (\text{A.12})$$

In terms of  $\rho$ , this is

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p'}{\partial z} - g \frac{\frac{1}{\rho} - \frac{1}{\rho_0}}{\frac{1}{\rho_0}} = D_w \quad (\text{A.13})$$

which is

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p'}{\partial z} + g \frac{\rho'}{\rho} = D_w \quad (\text{A.14})$$

This can be expressed in terms of temperature and pressure perturbations for the buoyancy term because

$$-\frac{\rho'}{\rho} = \frac{\rho_0}{\rho} - 1 = \frac{p_0 T}{p T_0} - 1 = \frac{p_0}{p} \left( \frac{T}{T_0} - \frac{p}{p_0} \right) = \frac{p_0}{p} \left( \frac{T'}{T_0} - \frac{p'}{p_0} \right) \quad (\text{A.15})$$

So

$$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p'}{\partial z} - g \frac{p_0}{p} \left( \frac{T'}{T_0} - \frac{p'}{p_0} \right) = D_w \quad (\text{A.16})$$

## A.4 Coordinate Transformation

General coordinate transformation  $(x, y, z) \rightarrow (x, y, \sigma)$

$$\left( \frac{\partial}{\partial x} \right)_z \rightarrow \left( \frac{\partial}{\partial x} \right)_\sigma - \left( \frac{\partial z}{\partial x} \right)_\sigma \frac{\partial}{\partial z} \quad (\text{A.17})$$

but  $\delta z = \frac{-\delta p_0}{\rho_0 g} = -\frac{(p^* \delta \sigma + \sigma \delta p^*)}{\rho_0 g}$ , so

$$\left( \frac{\partial}{\partial x} \right)_z \rightarrow \left( \frac{\partial}{\partial x} \right)_\sigma - \frac{\sigma}{p^*} \frac{\partial p^*}{\partial x} \frac{\partial}{\partial \sigma} \quad (\text{A.18})$$

## A.5 Derivation of $\dot{\sigma}$ Relation

$$\sigma = \frac{p_0 - p_{top}}{p_{surf} - p_{top}} = \frac{p_0 - p_{top}}{p^*} \quad (\text{A.19})$$

where  $p_{top}$  and  $p_{surf}$  are the values of  $p_0$  at the top and surface and  $p^* = p_{surf} - p_{top}$ .

$$\dot{\sigma} = \frac{D\sigma}{Dt} \quad (\text{A.20})$$

Therefore

$$\dot{\sigma} = \frac{1}{p^*} \frac{Dp_0}{Dt} - \frac{(p_0 - p_{top}) Dp^*}{(p^*)^2 Dt} \quad (\text{A.21})$$

Expanding total derivatives noting that  $p_0 = p_0(z)$  and  $p^* = p^*(x,y)$  and also that  $p_0$  is hydrostatic

$$\dot{\sigma} = -\frac{\rho_0 g}{p^*} w - \frac{\sigma}{p^*} \left( u \frac{\partial p^*}{\partial x} + v \frac{\partial p^*}{\partial y} \right) \quad (\text{A.22})$$